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An explicit solution of third-order difference equations

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Abstract

The solutions of homogeneous and nonhomogeneous third-order linear difference equations are obtained.

Keywords: Explicit solution; Difference equations

1. Introduction

The theory of difference equations has become significant for its various applications in numerical analysis. Several monographs have been devoted to the theory; however, most of them are generally concerned with the methods of solution of particular equations. In [3], Popenda has obtained an expression for the solutions of second-order difference equations. The present author [1,2] obtained solutions of certain linear difference equations and gave necessary and sufficient criteria for the exponential growth of the solution of these equations. In this note we obtain an explicit formula for the solutions of homogeneous and nonhomogeneous third-order linear difference equations. Before giving the main results, we shall introduce some notation which is needed in the discussion.

Let N be the set $\{1, 2, 3,\dots\}$. The expression $\sum_{i=1}^{t-1} a(i) + c_1$ is the solution of the linear difference equation $\Delta y = a(t)$ with the initial condition $Y(1) = C_1$, for all $t \in N$. Here Δ is the forward difference operator:

$$\Delta y = y(t+1) - y(t).$$

The expression $\prod_{i=1}^{t-1} b(i) c_1$ is the solution of the linear difference equation

$$y(t+1) = b(t) y(t), \quad \text{for all } t \in N,$$

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with the initial condition $y(1)=c_1$, for all $t \in N$. Here it is assumed that

$$\prod_{i=1}^{0} b(1)=1.$$

We make the following definitions:

$$\left[\sum_{m,n} a_i\right]=1+\sum_{i=m}^{n} a(i)+\sum_{i=m}^{n-3} a(i)\sum_{i_1=i+3}^{n} a(i_1)+\cdots+\sum_{i=m}^{m+\alpha} a(i)\sum_{i_1=i+3}^{m+\alpha+3} a(i_1)\sum_{i_k=i_{k-1}+3}^{n} a(i_k), \tag{1.1}$$

$k=\frac{1}{3}(n-(m+\alpha))$ and $m=3$, 4, 5, n is a positive integer and $\alpha=1$, 2.

If $\alpha=0$, the last term in the above relation becomes $a(m)a(m+3)a(m+6)\cdots a(n)$.

$$\left[\sum_{m,\alpha} a_i\right]=\begin{cases}1, & \text{if } 1 \leqslant m-\alpha \leqslant 3,\\ 0, & \text{if } m-\alpha \geqslant 4.\end{cases} \tag{1.2}$$

In case $\alpha=m$, $[\Sigma_{m,m} a_i]=((1+a(m))$.

In the present work, we consider a linear difference equation

$$L\{y\}=f(t), \quad t \in N,$$

where

$$L\{y\}=y(t+3\delta)-y(t+2\delta)-a(t)y(t).$$

For simplicity and without loss of generality, let $\delta=1$. That is, we consider

$$y(t+3)=y(t+2)+a(t)y(t)+f(t),$$

with initial data

$$y(1)=c_1, \qquad y(2)=c_2, \qquad y(3)=c_3.$$

2. Main results

Theorem 1. *The solution of the difference equation*

$$y(t+3)=y(t+2)+a(t)y(t), \tag{2.1}$$

with initial data

$$y(1)=c_1, \qquad y(2)=c_2, \qquad y(3)=c_3, \tag{2.2}$$

can be represented by the formula

$$y(n)=\left[\sum_{3,n-3} a_i\right]c_3+\left[\sum_{4,n-3} a_i\right]a(1)c_1+\left[\sum_{5,n-3} a_i\right]a(2)c_2, \tag{2.3}$$

where $t=n$ (n *integer*, $n>5$).

Proof. Since

$$\left[\sum_{m,n} a_i\right] = \left[\sum_{m,n-1} a_i\right] + a(n)\left[\sum_{m,n-3} a_i\right], \quad m = 3, 4, 5, \tag{2.4}$$

comparing the relation (2.4) with (2.1), we have that

$$\left[\sum_{3,n-3} a_i\right], \quad \left[\sum_{4,n-3} a_i\right], \quad \left[\sum_{5,n-3} a_i\right]$$

are solutions of (2.1).

Since these solutions are linearly independent, the general solution has the form

$$y(n) = \left[\sum_{3,n-3} a_i\right] d_1 + \left[\sum_{4,n-3} a_i\right] d_2 + \left[\sum_{5,n-3} a_i\right] d_3.$$

It is easy to show that

$$d_1 = c_3, \qquad d_2 = a(1)c_1, \qquad d_3 = a(2)c_2. \qquad \square$$

Now we will consider

$$x(n+3) = ax(n+2) + b(n)x(n), \quad a > 0. \tag{2.5}$$

Dividing (2.5) by a^{n+3}, it reduces to

$$y(n+3) = y(n+2) + a(n)y(n),$$

where

$$y(n) = a^{-n}x(n) \quad \text{and} \quad a(n) = a^{-3}b(n). \tag{2.6}$$

The solution of (2.5) is

$$x(n) = a^n y(n),$$

where $y(n)$ is defined in (2.3) and $a(n)$ in (2.6). Consider the difference equation

$$x(n+3) = q_1(n)x(n+2) + q_2(n)x(n). \tag{2.7}$$

After dividing (2.7) by $\prod_{i=1}^{n} q_1(i)$, we have

$$y(n+3) = y(n+2) + a(n)y(n),$$

where

$$y(n) = \frac{x(n)}{\prod_{i=1}^{n-3} q_1(i)}$$

and

$$a(n) = \frac{q_1(n)}{q_1(n)q_1(n-1)q_1(n-2)}, \quad \prod_{i=0}^{-2} q_1(i) = 1. \tag{2.8}$$

The solution of (2.7) is

$$x(n)=\prod_{i=0}^{n-3} q_1(i)y(n),$$

where $y(n)$ and $a(n)$ are defined in (2.3) and (2.8) respectively.

Let us consider the initial problem

$$y(n+3)=y(n+2)+a(n)y(n)+f(n), \tag{2.9}$$

$$y(i)=c_i, \quad i=1,2,3, \tag{2.10}$$

c_i, $i=1, 2, 3$, are constants, $a(n)$ and $f(n)$ are continuous functions.

Theorem 2. *The solution of* (2.9) *can be expressed in the form*

$$\begin{aligned} y(n)={}&\Big[\sum_{3,n-3} a_i\Big]c_3+\Big[\sum_{4,n-3} a_i\Big]a(1)c_1+\Big[\sum_{5,n-3} a_i\Big]a(2)c_2\\ &+\sum_{i=1}^{n-6}\Big[\sum_{i+3,n-3} a_i\Big]f(i)+f(n-5)+f(n-4)+f(n-3), \quad n\geqslant 7. \end{aligned} \tag{2.11}$$

Proof. Since

$$\Big[\sum_{i+3,n} a_i\Big]=\Big[\sum_{i+3,n-1} a_i\Big]+a(n)\Big[\sum_{i+3,n-3} a_i\Big], \tag{2.12}$$

we have

$$\begin{aligned} \sum_{i=1}^{n-3}\Big[\sum_{i+3,n} a_i\Big]f(i)&=\sum_{i=1}^{n-3}\Big[\sum_{i+3,n-1} a_i\Big]f(i)+a(n)\sum_{i=1}^{n-3}\Big[\sum_{i+3,n-3} a_i\Big]f(i)\\ &=\sum_{i=1}^{n-4}\Big[\sum_{i+3,n-1} a_i\Big]f(i)+f(n-3)\\ &\quad+a(n)\left\{\sum_{i=1}^{n-6}\Big[\sum_{i+3,n-3} a_i\Big]f(i)+f(n-5)+f(n-4)+f(n-3)\right\}. \end{aligned} \tag{2.13}$$

Adding $f(n-2)+f(n-1)+f(n)$ termwise to (2.13), we have

$$\begin{aligned} &\sum_{i=1}^{n-3}\Big[\sum_{i+3,n} a_i\Big]f(i)+f(n-2)+f(n-1)+f(n)\\ &\quad=\sum_{i=1}^{n-4}\Big[\sum_{i+3,n-1} a_i\Big]f(i)+f(n-3)+f(n-2)+f(n-1)\\ &\qquad+a(n)\left\{\sum_{i=1}^{n-6}\Big[\sum_{i+3,n-3} a_i\Big]f(i)+f(n-5)+f(n-4)+f(n-3)\right\}+f(n). \end{aligned} \tag{2.14}$$

By adding

$$\left[\sum_{3,n-2j-1} a_i\right]c_3, \qquad \left[\sum_{3,n-2i-1} a_i\right]a(1)c_1, \qquad \left[\sum_{3,n-2j-1} a_i\right]a(2)c_2$$

to (2.14), $j = 0$, 1, the right-hand side of (2.14) becomes

$$y(n+2) + a(n)y(n) + f(n),$$

which is equal to $y(n+3)$. It follows that (2.11) is a solution of the equation

$$y(n+3) = y(n+2) + a(n)y(n) + f(n). \qquad \square$$

Remark.

$$y(n) = \left[\sum_{3,n-2} a'_i\right](a(1)y(1) + y(2)) + \left[\sum_{4,n-2} a'_i\right]a(2)y(2) + \sum_{i=1}^{n-4}\left[\sum_{i+2,n-2} a'_i\right]f(i) + f(n-3) + f(n-2)$$

is a solution of the difference equation

$$y(n+2) = y(n+1) + a(n)y(n) + f(n),$$

where

$$\left[\sum_{3,n} a'_i\right] = \left\{1 + \sum_{i=3}^{n} a(i) + \sum_{i=3}^{n-2} a(i) \sum_{i_1=i+2}^{n} a(i_1) + \cdots + \sum_{i=3}^{5} a(i) \cdots \sum^{n} \tfrac{1}{2}a(i_{n-5}) + a_3 a_5 \cdots a_n\right\}$$

if n is odd,

$$\left[\sum_{3,n} a'_i\right] = \left\{1 + \sum_{i=3}^{n} a(i) + \sum_{i=3}^{n-2} a(i) \sum_{i_1=i+2}^{n} a(i_1) + \cdots + \sum_{i=3}^{4} a(i) \sum_{i_1=i+2}^{n} a(i_1) \cdots \sum^{n} \tfrac{1}{2}a(i_{n-4})\right\}$$

if n is even.

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